The Casimir Energy in a Separable Potential

by

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ABSTRACT

Abstract

The Casimir energy is the first-order-in-$\hbar$ correction to the energy of a time-independent field configuration in a quantum field theory. We study the Casimir energy in a toy model, where the classical field is replaced by a separable potential. In this model the exact answer is trivial to compute, making it a good place to examine subtleties of the problem. We construct two traditional representations of the Casimir energy, one from the Greens function, the other from the phase shifts, and apply them to this case. We show that the two representations are correct and equivalent in this model. We study the convergence of the Born approximation to the Casimir energy and relate our findings to computational issues that arise in more realistic models.

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1 Introduction

Casimir energies are important in quantum field theories, where they give the first-order (in $\hbar$) corrections to the energy of time-independent classical field configurations. Computations of the Casimir energy involve formal manipulations of divergent expressions that eventually are regulated and renormalized.[2] Since they go beyond perturbation theory, Casimir energy calculations provide a potentially powerful way to study nonperturbative effects. Several formally equivalent representations of the Casimir energy have been used in numerical calculations, notably (a) in terms of integrals over scattering phase shifts and bound states; (b) as an integral over the trace of a Greens function (equivalent to a functional determinant); and (c) as an infinite sum of Feynman diagrams. The aim of this paper is to study the various representations of the Casimir energy in a simple, highly convergent toy model, where the equivalence of different representations can be demonstrated without the complication of divergences. Our analysis does not demonstrate the equivalence of representations in realistic models, where divergences make the arguments more complex. However, we do see how the different representations are related in a simple, calculable example. Also, because the model is so simple, it is possible to explore some issues of convergence and cancellation that are obscure in more realistic cases.

The model we study makes use of the simple dynamics of separable potentials in nonrelativistic quantum mechanics.[1] Our first task is to explain how such a simple model can give insight into one-loop effects in a quantum field theory. Next we review the well-known Greens function and scattering theory analysis of a separable potential. With this in hand we discuss the various representations of the Casimir energy and explore their relationship. Finally, we study convergence of the perturbative expansion and associated computational issues.

Formally, the Casimir energy is the sum of zero point energies for the modes of a quantum field, Ψ, in the presence of some spatially varying background, $\phi_0(x)$,

$$\mathcal{E}_c = \tfrac{1}{2}\sum(\hbar\omega[\phi_0] - \hbar\omega_0) \tag{1}$$

where $\{\omega[\phi_0]\}$ are the eigenfrequencies in the presence of the nontrivial background field ϕ_0, and $\{\omega_0\}$ are the eigenfrequencies in the trivial background, $\phi_0 = 0$. The $\{\omega[\phi_0]\}$ are typically related to the eigenvalues of some simple differential operator, $\mathcal{H}[\phi_0]$, which looks like a Schrödinger operator for a scalar field,

$$\mathcal{H} = -\frac{d^2}{dx^2} + V_{\phi_0}(x) \tag{2}$$

or a Dirac operator for a spinor field. $V_{\phi_0}(x)$ is a "potential" derived from the field ϕ_0. Although we have written eq. (2) for a scalar in one-dimension, the same considerations apply in three-dimensions. Typically the $\{\omega\}$ are

not themselves the eigenvalues, $\{\lambda\}$, of $\mathcal{H}$ but rather simple functions of the them. For example, in the case that Ψ is a scalar, $\omega_j = \sqrt{\lambda_j + m^2}$.

Our toy model is based on two alterations in this physical picture: first we assume that the $\{\omega\}$ are proportional to the eigenvalues of $\mathcal{H}$, as they would be in the nonrelativistic case where $\hbar\omega(k) = \hbar^2 k^2/2m$ (and k^2 is the eigenvalue of $\mathcal{H}$); and second, we replace the *local* potential, $V(x)$, by a nonlocal, but *separable* potential,

$$V(x, x') = -\lambda u(x) u(x') \tag{3}$$

where $\int dx\, u^2(x) = 1$. Because of our first assumption the Casimir energy can be written as

$$\mathcal{E}_c = \tfrac{1}{2}\,\mathrm{Tr}(\mathcal{H} - \mathcal{H}_0)\ . \tag{4}$$

As a consequence of the separability assumption, $\mathcal{H}$ can be written formally as

$$\mathcal{H} = \mathcal{H}_0 - \lambda |u\rangle\langle u| \tag{5}$$

where $|u\rangle$ is the state with wavefunction $u(x) = \langle x|u\rangle$. Combining eqs. (4) and (5) we find that the Casimir energy is $-\frac{1}{2}\lambda$,

$$\begin{aligned} \mathcal{E}_c &= -\tfrac{1}{2}\,\mathrm{Tr}\{-\lambda |u\rangle\ \langle u|\} \\ &= -\tfrac{1}{2}\lambda\ . \end{aligned} \tag{6}$$

This is the fundamental result that makes the study of the Casimir energy in separable potential models interesting – the answer is so transparent. Our object in this paper is to show that more conventional (and more complicated) methods of computing the Casimir energy coincide with this simple result.

The remainder of the paper is organized as follows. In the next section we review the bound states, Greens functions and scattering amplitude for a separable potential. In Section 3 we derive expressions for the Casimir energy in terms of integrals over Greens functions or scattering phase shifts. In Section 4 we compute the Casimir energy using the methods of Section 3 for the case of a separable potential and show that the result is $-\frac{1}{2}\lambda$. In Section 5 we specialize to a particular choice of u and explore the convergence of the Casimir energy calculation as a function of the strength, λ, of the interaction. We conclude in Section 6.

2 Separable Potentials

In this section we review the solution of the scattering problem for a scalar particle moving in an s-wave separable potential in three dimensions. We

begin with the Schrödinger equation in three dimensions with a nonlocal potential,

$$-\frac{\hbar^2}{2M}\nabla^2\psi(\vec{r}) + \int d^3r'\, V(\vec{r},\vec{r}')\psi(\vec{r}') = \omega\psi(\vec{r}) \tag{7}$$

where

$$\langle\vec{r}|V|\vec{r}'\rangle = V(\vec{r},\vec{r}')\ . \tag{8}$$

The resulting integral equation is in general more complicated than the local case, but for a separable potential, eq. (3), it is simpler. For simplicity we take a *separable and spherically symmetric* nonlocal potential,

$$V(\vec{r},\vec{r}') = -\lambda u(r)u(r') \tag{9}$$

for which eq. (7) reduces to

$$-\frac{\hbar^2}{2M}\nabla^2\psi(\vec{r}) - \lambda u(r)\int d^3r'\, u(r')\psi(\vec{r}') = \omega\psi(\vec{r})\ . \tag{10}$$

For $\lambda > 0$ the potential is attractive.

A principal simplification with a spherically symmetric, separable potential is the absence of any interaction in partial waves with $\ell > 0$. This follows immediately from eq. (10) because $\int d^3r'\, u(r')\psi(\vec{r}')$ projects out only the spherically symmetric part of ψ. All partial waves except $\ell = 0$ cancel out of the Casimir sum. We replace $\psi(\vec{r})$ by $U(r)/r$ and find that eq. (10) simplifies to

$$-U''(r) - 4\pi\lambda r u(r)\int_0^\infty dr' r' u(r')U(r') = k^2U(r) \tag{11}$$

where $k^2 = 2M\omega/\hbar^2$, and where we have set $\hbar = 2M = 1$ henceforth.

For scattering solutions, it is useful to convert eq. (11) into an integral equation for the function, $U^{(+)}(r)$, obeying scattering boundary conditions,

$$U^{(+)}(r) = U_0(r) + 4\pi\lambda\int_0^\infty dr' G_0^{(+)}(r,r',k)r'u(r')\int_0^\infty dr'' r'' u(r'')U^{(+)}(r'') \tag{12}$$

where $U_0(r)$ is the free solution regular at the origin, $U_0(r) = \sin kr$, and $G_0^{(+)}(r,r',k)$ is the free, s-wave Greens function with outgoing wave boundary conditions,

$$\begin{aligned} G_0^{(+)}(r,r',k) &= \frac{1}{k}e^{ikr_>}\sin kr_< \\ &= \frac{1}{\pi i}\int_{-\infty}^{\infty} dq\, \frac{e^{iqr}\sin qr'}{(q^2 - k^2 - i\epsilon)}\ . \end{aligned} \tag{13}$$

The properties of the Greens function guarantee that $U^{(+)}(r)$ satisfies the Schrödinger equation. The asymptotic form of the scattering wave at large r,

$$\lim_{r\to\infty} U^{(+)}(r) = \frac{i}{2}(e^{-ikr} - e^{2i\delta(k)}e^{ikr}) \tag{14}$$

and the behavior of the Greens function at large r enable us to read off the scattering amplitude in terms of U,

$$f(k) \equiv e^{i\delta(k)}\sin\delta(k) = \frac{4\pi\lambda}{k}\int_0^\infty dr\, r\sin kru(r)\int_0^\infty dr'\, r'U^{(+)}(r')u(r')\ . \tag{15}$$

To proceed we return to eq. (12), multiply by $ru(r)$ and integrate over r. The resulting algebraic equation can be solved for $\int_0^\infty dr\, ru(r)U^{(+)}(r)$, which yields $f(k)$ upon substitution into eq. (15),

$$f(k) = \frac{4\pi\lambda}{k}\frac{|\xi_0(k)|^2}{1-X(\omega)} \tag{16}$$

where we have defined

$$\begin{aligned} \xi_0(k) &\equiv \int_0^\infty dr\, ru(r)U_0(r) = \int_0^\infty dr\, r\sin kru(r) \\ X(\omega) &= 8\lambda\int_0^\infty dq\,\frac{1}{q^2-\omega-i\epsilon}|\xi_0(q)|^2\ . \end{aligned} \tag{17}$$

The scattering amplitude, $f(k)$, has a cut along the positive k^2 axis induced by the cut in $X(\omega)$. The cut begins with a branch point at threshold, $k^2 = 0$. The discontinuity across the cut is given by

$$\mathrm{disc}\, f(k) = 2i\,\mathrm{Im}\, f(k) = 2i\sin^2\delta(k) \tag{18}$$

where we have used $1/(x+i\epsilon) = \mathrm{PV}(1/x) - i\pi\delta(x)$ to separate out the imaginary part when x is real. The phase shift $\delta(k)$ can be read off eq. (16) conveniently by using the parameterization, $f(k) = 1/(\cot\delta - i)$,

$$\tan\delta(k) = \frac{4\pi\lambda|\xi_0(k)|^2/k}{1-8\lambda\int_0^\infty dq\,|\xi_0(q)|^2/(q^2-k^2)}\ . \tag{19}$$

The Born approximation to the scattering amplitude is obtained by expanding the denominator in eq. (16) in a geometric series,

$$f_{\mathrm{BA}}(k) = \frac{4\pi\lambda|\xi_0(k)|^2}{k}\sum_{n=0}^\infty\left[8\lambda\int_0^\infty dq\,\frac{|\xi_0(q)|^2}{q^2-k^2-i\epsilon}\right]^n\ . \tag{20}$$

In addition to the branch cut for real, positive k^2, $f(k)$ can be singular at values of k where the denominator in eq. (16) vanishes, i.e., where $X(\omega) = 1$. Bound states appear as poles in $f(k)$ for $k^2 < 0$, or more precisely, $k = i\kappa$. [Poles at $k = -i\kappa$ are "virtual states".] When $k^2 < 0$, $X(-\kappa^2)$ becomes real,

$$X(-\kappa^2) = 8\lambda \int_0^\infty dq \, \frac{|\xi_0(q)|^2}{q^2 + \kappa^2} \tag{21}$$

and a bound state occurs at the value of $\kappa \equiv \kappa_0$ for which

$$X(-\kappa_0^2) = 8\lambda \int_0^\infty dq \, \frac{|\xi_0(q)|^2}{q^2 + \kappa_0^2} = 1 \ . \tag{22}$$

Since $X(-\kappa^2)$ is a decreasing function of κ, the criterion for existence of a bound state is that $X(0) > 1$, or

$$8\lambda \int_0^\infty dq \, \frac{|\xi_0(q)|^2}{q^2} > 1 \ . \tag{23}$$

This equation defines the critical value of λ at which a bound state appears for a given choice of $u(r)$. Note that there is at most one bound state in this separable potential. This completes our review of the properties of a particle moving in a separable s-wave potential.

3 The Casimir Energy

We are interested in (a) the Greens function and (b) the phase shift expressions for the Casimir energy. Both representations may be derived heuristically in the effective action formalism in field theory.[2] Equally well, we can start from the formal expression, eq. (1), and convert the sum over eigenenergies to a trace of a Greens function or an integral over phase shifts.

3.1 Greens Function Representation

The Greens function method starts from the Greens function in coordinate/energy representation,

$$G(\vec{r}, \vec{r}', \omega) = \sum_j \frac{\phi_j(\vec{r})\phi_j^*(\vec{r}')}{\omega_j - \omega} \tag{24}$$

where the $\{\phi_j(\vec{r})\}$ are the unit-normalized* energy eigenstates in coordinate space and the summation ranges over the spectrum of $\mathcal{H}$, so

$$(\mathcal{H} - \omega)G(\vec{r}, \vec{r}', \omega) = \delta^3(\vec{r} - \vec{r}') \ . \tag{25}$$

In order to relate this to the Casimir energy, consider the difference of the Greens functions evaluated at $\omega - i\epsilon$ and $\omega + i\epsilon$ integrated over space,

$$\int_0^\infty d^3r \, [\Delta G(\vec{r}, \vec{r}, \omega + i\epsilon) - \Delta G(\vec{r}, \vec{r}, \omega - i\epsilon)] = 2\pi i \sum_j [\delta(\omega - \omega_j) - \delta(\omega - \omega_{0j})] \tag{26}$$

where ΔG is the difference between the Greens function in the background potential and the free Greens function. To obtain the Casimir energy, multiply eq. (26) by ω, integrate from $-\infty$ to ∞, and divide by $4\pi i$,

$$\mathcal{E}_c = \frac{1}{4\pi i} \int_{-\infty}^{\infty} d\omega \, \omega \int_0^\infty d^3r \, [\Delta G(\vec{r}, \vec{r}', \omega + i\epsilon) - \Delta G(\vec{r}, \vec{r}', \omega - i\epsilon)] \ . \tag{27}$$

This result can be simplified further by introducing a function, $F(\omega)$, defined by its derivative,

$$\frac{dF(\omega)}{d\omega} = \int d^3r \, \Delta G(\vec{r}, \vec{r}, \omega) \tag{28}$$

and the condition that $F(\omega) \to 0$ as $|\omega| \to \infty$. Then substituting into eq. (27) and integrating by parts, we find

$$\mathcal{E}_c = \frac{1}{4\pi i} \int_{-\infty}^{\infty} d\omega \, [F(\omega - i\epsilon) - F(\omega + i\epsilon)] \ . \tag{29}$$

The surface terms at $\pm\infty \pm i\epsilon$ can be shown to cancel.

Referring back to the definition of ΔG, we see that the discontinuity in F along the Re ω axis is associated with the eigenstates of $\mathcal{H}$. Let ω_0 be the ground state of $\mathcal{H}$. For all $\omega < \omega_0$ we can send $\epsilon \to 0$. Then the integral of eq. (29) can be written as a contour integral over a contour $\mathcal{C}$ (as shown in Fig. 1) that runs in from $+\infty$ above the Re ω axis to ω_0 and then returns to $+\infty$ below the Re ω axis,

$$\mathcal{E}_c = \frac{1}{4\pi i} \oint_{\mathcal{C}} d\omega \, F(\omega) = \frac{1}{4\pi i} \oint_{\mathcal{C}_\infty} d\omega \, F(\omega) \ . \tag{30}$$

In the second term we have replaced the contour $\mathcal{C}$ by the counterclockwise circle at infinity, $\mathcal{C}_\infty$, since in general the only singularities in $F(\omega)$ lie on the real axis. This compact expression will be particularly useful in the separable case.

*We proceed as if the spectrum is discrete and the states are normalizable. Our result applies equally to the case (of interest) where the spectrum is continuous and the states are normalized to δ-functions.

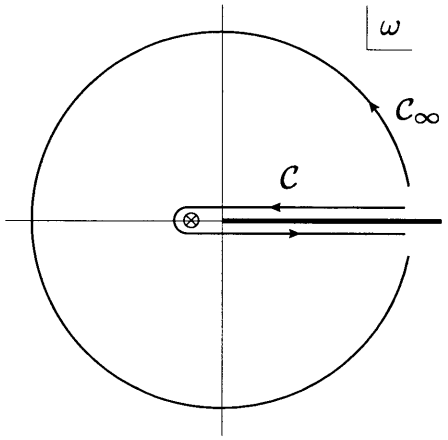

Figure 1: Contours in the complex ω-plane relevant to evaluating the Casimir energy.

3.2 Phase Shift Representation

The phase shift representation for $\mathcal{E}_c$ begins from eq. (1) divided up into bound state and continuum contributions,

$$\begin{aligned} \mathcal{E}_c &= \tfrac{1}{2} \sum_{\substack{\text{bound} \\ \text{states}}} \omega_j + \tfrac{1}{2} \int_{\text{continuum}} dn\, \omega \\ &= -\tfrac{1}{2} \sum_j B_j + \tfrac{1}{2} \int_0^\infty d\omega\, \rho(\omega)\omega \end{aligned} \tag{31}$$

where B_j are the binding energies, and $\rho(\omega) \equiv dn/d\omega$ is the *modification* of the continuum density of states due to the interaction. A simple and general argument connects $\rho(\omega)$ to the phase shift (in our case only the s-wave contributes),

$$\rho(\omega) = \frac{1}{\pi} \frac{d\delta(\omega)}{d\omega} \; . \tag{32}$$

Substituting $\rho(\omega)$, and integrating by parts yields,

$$\mathcal{E}_c = -\tfrac{1}{2} \sum_j B_j - \frac{1}{2\pi} \int_0^\infty d\omega\, \delta(\omega) \; . \tag{33}$$

The surface term in the integration by parts $\sim \omega\delta(\omega)$ vanishes trivially at $\omega = 0$. For a general problem the surface term at infinity causes problems

because generically $\delta(\omega) \sim 1/\omega$ as $\omega \to \infty$. However, in the case of a separable potential, we shall see that the phase shift vanishes rapidly as $\omega \to \infty$, so the surface term can be ignored. So we may immediately calculate the Casimir energy if we are given the phase shift. Eq. (30) and eq. (33) give alternative representations of the Casimir energy, which should both be equivalent to eq. (6) for the separable case.

4 Casimir Energy for the Separable Case

In this section we examine the Green's function and phase shift representations of the Casimir energy for the separable case.

4.1 Greens Function Representation

To exploit eq. (27) we need an explicit expression for the trace of the Greens function for the separable potential of Section 2. We start with the Lippmann-Schwinger equation for G,

$$G(\vec{r},\vec{r}',\omega) = G_0(\vec{r},\vec{r}',\omega) + \lambda \int d^3r_1\, d^3r_2\, G_0(\vec{r},\vec{r}_1,\omega)u(r_1)u(r_2)G(\vec{r}_2,\vec{r}',\omega) \tag{34}$$

which may be solved by iteration,

$$\Delta G(\vec{r},\vec{r}',\omega) = \lambda \int d^3r_1\, d^3r_2\, G_0(\vec{r},\vec{r}_1,\omega)u(\vec{r}) \left[1+X(\omega)+X(\omega)^2+\cdots\right] u(\vec{r}')G_0(\vec{r}_2,\vec{r}',\omega) \tag{35}$$

where $X(\omega)$ was defined in eq. (17).

Eq. (27) requires the us to set $\vec{r} = \vec{r}'$ and integrate. If we substitute from eq. (35) the result takes a particularly simple form,

$$\begin{aligned} \int d^3r\, \Delta G(\vec{r},\vec{r},\omega) &= \frac{dX}{d\omega} \sum_{m=0}^{\infty} \frac{[X(\omega)]^m}{m} \\ &= -\frac{d}{d\omega} \ln\left(1 - X(\omega)\right) \end{aligned} \tag{36}$$

so

$$\mathcal{E}_C = -\frac{1}{4\pi i} \oint_{\mathcal{C}_\infty} d\omega\, \ln\left(1 - X(\omega)\right) \tag{37}$$

where we have identified the generic function $F(\omega)$ of eq. (28) with $-\ln(1 - X(\omega))$ in the separable case. [With this identification it is easy to check that the surface terms discussed following eq. (29) do indeed vanish.] To

evaluate the integral we need $\lim_{\omega\to\infty} \ln(1 - X(\omega))$. From eq. (17) and the normalization of $u(r)$, it is easy to see that as $\omega \to \infty$,

$$\ln(1 - X(\omega)) \sim -X(\omega) \sim \lambda/\omega \; . \tag{38}$$

Thus,

$$\mathcal{E}_c = -\frac{\lambda}{4\pi i} \oint_{C_\infty} \frac{d\omega}{\omega} = -\frac{\lambda}{2} \; . \tag{39}$$

Confirming the value of the Casimir energy is $-\lambda/2$ as we derived in Section 1 from more formal considerations.

Some comments are in order:

- The method we have presented is equivalent to the usual, graphical analysis of the effective action in quantum field theory. Had we begun with a theory describing a scalar field, Ψ, propagating in a scalar background, Φ, then the Casimir energy would have been given by the (infinite) sum of one-loop graphs for Ψ with insertions of the Ψ/Φ coupling. The one-loop graphs could then be resummed into a Lippmann Schwinger equation for $G(\vec{r}, \vec{r}', \omega)$. The special advantages of a separable potential are (a) that the resulting integral equation for G is solvable, and (b) that there are no ultraviolet divergences (seen as divergences in the $\omega \to \infty$ limit) to complicate the calculation.

- The resummation of $1 + X + X^2 + \cdots$ that generated the logarithm in eq. (36) is valid for small λ where the series converges. The result can then be analytically continued to large λ were the re-expansion into a (Born) series does not converge.

4.2 Phase Shift Representation

In Section 3 we also derived a simple representation, eq. (33), for the Casimir energy as a sum over binding energies and an integral over the phase shift $\delta(\omega)$. In this subsection we show explicitly that this representation is equivalent to the Green's function representation as found, for example, in eq. (37).

First, we return to the contour, $\mathcal{C}$, of Fig. 1,

$$\mathcal{E}_c = -\frac{1}{4\pi i} \oint_{\mathcal{C}} d\omega \, \ln(1 - X(\omega)). \tag{40}$$

The contour integral is equivalent to integrating the discontinuity in the integrand across the real axis from just below the lowest bound state ω_0 to ∞,

$$\mathcal{E}_c = \frac{1}{2\pi} \int_{\omega_0}^{\infty} d\omega \, \mathrm{Im} \ln(1 - X(\omega + i\epsilon)) \tag{41}$$

where we have used the fact that the discontinuity in the logarithm is $2i$ times its imaginary part.

There are two distinct regions in the integral. For $\omega < 0$, $X(\omega)$ is explicitly real, and the logarithm can have an imaginary part if and only if $X(\omega) > 1$. For $\omega > 0$, $X(\omega)$ always has an imaginary part (see eq. (17)) related to the scattering amplitude. We shall show that these two contributions map into the binding energy and integral over the phase shift respectively, as expected from eq. (33).

First consider $\omega \leq 0$. According to the analysis of Section 2, eqs. (21) and (22), etc., $X(\omega)$ takes its maximum at $\omega = 0$, and decreases as ω decreases. If $X(0) > 1$ there is a single bound state with binding energy, $B \equiv -\omega_0$, determined by the equation $X(-B) = 1$. Thus if λ is large enough to generate a bound state, then $\ln(1 - X(\omega))$ has an imaginary part (equal to π) for $-B \leq \omega \leq 0$. Thus

$$\frac{1}{2\pi}\int_{\omega_0}^{0} d\omega \,\mathrm{Im}\ln(1 - X(\omega)) = -\tfrac{1}{2}B \ . \tag{42}$$

Next consider $\omega > 0$. From the analysis of Section 2, we find

$$\ln(1 - X(\omega)) = \ln\left(1 - 4\lambda \fint_{-\infty}^{\infty} dq\, \frac{|\xi_0(q)|^2}{(q^2 - \omega)} + 4\lambda i\pi \frac{|\xi_0(k)|^2}{k}\right) \ . \tag{43}$$

From this we read off the imaginary part,

$$\mathrm{Im}\ln(1 - X(\omega + i\epsilon)) = -\tan^{-1}\left(\frac{4\pi\lambda|\xi_0(k)|^2/k}{1 - 4\lambda\int_0^\infty dq\,|\xi_0(q)|^2/(q^2 - \omega)}\right) \tag{44}$$

which is just $-\tan\delta(k)$ as defined in eq. (19). Substituting from eqs. (42) and (44) into eq. (41), we confirm the phase shift plus binding energy representation, eq. (33),

$$\mathcal{E}_c = -\tfrac{1}{2}\sum_j B_j - \frac{1}{2\pi}\int_0^\infty d\omega\,\delta(\omega) \ . \tag{45}$$

5 Convergence and Numerical Issues in a Separable Potential Model

A particular virtue of the separable potential model is that it is simple enough to allow us to investigate issues that are difficult to attack in more realistic

theories. As an example of such an issue, we consider here some computational aspects of phase shift representation of the Casimir energy. Ref. [2] introduced a method of computing the Casimir energy in realistic quantum field theories based on the Born approximation to the phase shift. Divergences generated by the lowest-order Feynman diagrams are associated with the first few terms in the Born approximation to the phase shift. To remove the divergences from the numerical part of the calculation, the first few Born approximations are subtracted from the phase shift. These contributions are then added back into the calculation in the form of (divergent) Feynman diagrams, which are regulated and renormalized by means of available counterterms. Schematically

$$\mathcal{E}_c = -\tfrac{1}{2}\sum_j B_j - \frac{1}{2\pi}\int_0^\infty d\omega\, \bar{\delta}^N(\omega) + \sum_{n=0}^{N} D_n + \mathrm{CT} \tag{46}$$

where $\bar{\delta}^N$ is the phase shift with the first N terms in the Born approximation subtracted,

$$\bar{\delta}(\omega) = \delta(\omega) - \sum_{n=1}^{N} \lambda^n \delta^{(n)}(\omega) \tag{47}$$

D_n is the contribution to $\mathcal{E}_c$ from the n^{th}-order Feynman diagram, and CT denotes the renormalization counterterms. Both the $\{D_n\}$ and the counterterms depend on a cutoff if the theory has ultraviolet divergences. The Born-subtracted phase shift, however, is cutoff independent.

In realistic quantum field theory applications the Feynman diagrams, $\{D_n\}$, and the counterterms are calculated by standard, diagrammatic methods. The integral over $\bar{\delta}^N$ is done numerically. The Born approximations $\{\lambda^n\delta^{(n)}\}$ have just the right behavior to cancel the large ω tail of $\delta(\omega)$ and render the ω-integral convergent. However they are a poor approximation to $\delta(\omega)$ for small ω, especially in partial waves where there are bound states.[3] Numerical studies show that at low ω the function $\bar{\delta}^N(\omega)$ is much larger than its integral $\int_0^\infty d\omega\, \bar{\delta}^N(\omega)$. The origin of this effect is obscured in those studies because the integral over the individual Born approximations, $\int_0^\infty d\omega\, \delta^{(n)}(\omega)$, diverge. The large ω behavior of the separable potential model is very soft, enabling us to study this issue explicitly.

It is convenient to specialize further to a particular choice of the separable potential function, $u(r)$. Following Gottfried, we choose the Yukawa function,

$$u(r) = \sqrt{\frac{\alpha}{2\pi}}\frac{e^{-\alpha r}}{r}\ . \tag{48}$$

A simple calculation gives

$$\xi_0(k) \;=\; \sqrt{\frac{\alpha}{2\pi}}\frac{k}{k^2+\alpha^2}$$

$$X(k) \;=\; \frac{\lambda}{2(\alpha - ik)^2} \;. \tag{49}$$

It is convenient to use the momentum, k, rather than $\omega = k^2$ as the independent variable, also to introduce scaled variables, $k/\alpha \equiv z$, $\lambda/\alpha^2 \equiv g$, and finally, it is easiest to explore the model by studying the scattering amplitude, $f(k) \to f(g,z) = 1/(\cot\delta(g,z) - i)$,

$$f(g,z) = \frac{2gz}{(1+iz)^2((1-iz)^2 - g)} \;. \tag{50}$$

The scattering amplitude $f(g,z)$ has two types of singularities. The double pole at $z = -i$ is a "potential singularity" arising from the fourier transform of $u(r)$, $\xi_0(k)$. It is on the second sheet of the complex ω plane where it does not affect the contour shifting arguments we used earlier in this paper. The other singularities are poles at

$$z_\pm = \begin{cases} i(\sqrt{g} - 1) \\ -i(\sqrt{g} + 1). \end{cases} \;. \tag{51}$$

For $g > 1$, z_+ is associated with a bound state: it lies on the positive imaginary axis, corresponding to the negative real axis on the first sheet of the complex ω-plane.† The energy of the bound state is $\omega_0 = -\alpha^2 z_-^2 = -(\sqrt{\lambda} - \alpha)^2$. The pole at z_- is always on the second sheet and is not dynamically important. The singularities in the complex-z plane are summarized (for $g > 1$) in Fig. 2.

With these considerations in mind, we can write an illuminating expression for the Casimir energy,

$$\mathcal{E}_c = -\tfrac{1}{2}\lambda = \begin{cases} -\dfrac{1}{2\pi}\displaystyle\int_0^\infty d\omega\, \delta(\omega) & \text{for } \lambda < \alpha^2 \\ -\tfrac{1}{2}(\sqrt{\lambda} - \alpha)^2 - \dfrac{1}{2\pi}\displaystyle\int_0^\infty d\omega\, \delta(\omega) & \text{for } \lambda > \alpha^2 \end{cases} \tag{52}$$

i.e., where there is a bound state, it appears explicitly in $\mathcal{E}_c$. The Born approximation is an expansion of $\delta(g,z)$ in powers of g obtained by expanding the denominator of $f(g,z)$ (see eq. (50)) in a geometric series:

$$\delta(g,z) = \sum_{n=1}^{\infty} g^n \delta^{(n)}(z) \;. \tag{53}$$

†For $g < 1$ the z_+ pole lies on the negative imaginary axis, i.e., on the second sheet of the ω-plane. As $g \to 1$ it becomes virtual state on its way to becoming a bound state.

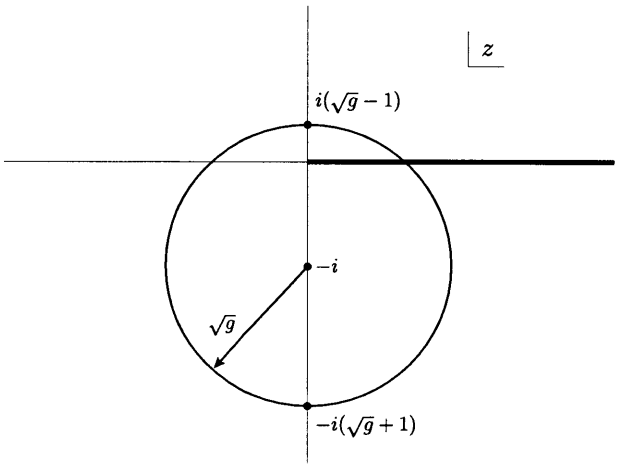

Figure 2: The Born expansion converges outside a circle of radius $\sqrt{g}$ centered at $-i$ in the complex z-plane. Singularities in $f(g,z)$ at $z=-i$, $z_+ = i(\sqrt{g}-1)$, and $z_- = -i(\sqrt{g}+1)$ are shown.

The convergence of the Born expansion is determined by the convergence of the geometric series for $f(g,z)$, which converges provided $|1-iz| > \sqrt{g}$. That is, z must lie outside the circle of radius $\sqrt{g}$ centered at $z=-i$. To compute the Casimir energy by the phase shift method, we must integrate over all $\omega > 0$, corresponding to $z \in [0,\infty]$.

As long as $g < 1$ ($\lambda < \alpha^2$) the Born expansion converges for all real, positive z and can be used under the integral sign in eq. (52) to compute $\mathcal{E}_c$,

$$\mathcal{E}_c = -\tfrac{1}{2}\lambda = -\frac{1}{2\pi}\sum_{n=1}^{\infty}\lambda^n \int_0^\infty d\omega\, \delta^{(n)}(\omega) \qquad \text{for } \lambda < \alpha^2 \ . \tag{54}$$

This is a remarkable result: both sides are power series in λ, but the left hand side consists of a single term. We conclude that, for $\lambda < \alpha^2$, when there is no bound state, the entire Casimir energy is given by the first Born approximation, and that the integrals over all higher Born approximations to the phase shift vanish:

$$\begin{aligned} \mathcal{E}_c &= -\tfrac{1}{2}\lambda = -\frac{\lambda}{2\pi}\int_0^\infty d\omega\, \delta^{(1)}(\omega), \\ 0 &= \int_0^\infty d\omega\, \delta^{(n)}(\omega) \qquad \text{for all } n > 1 \ . \end{aligned} \tag{55}$$

This result may easily be verified numerically.

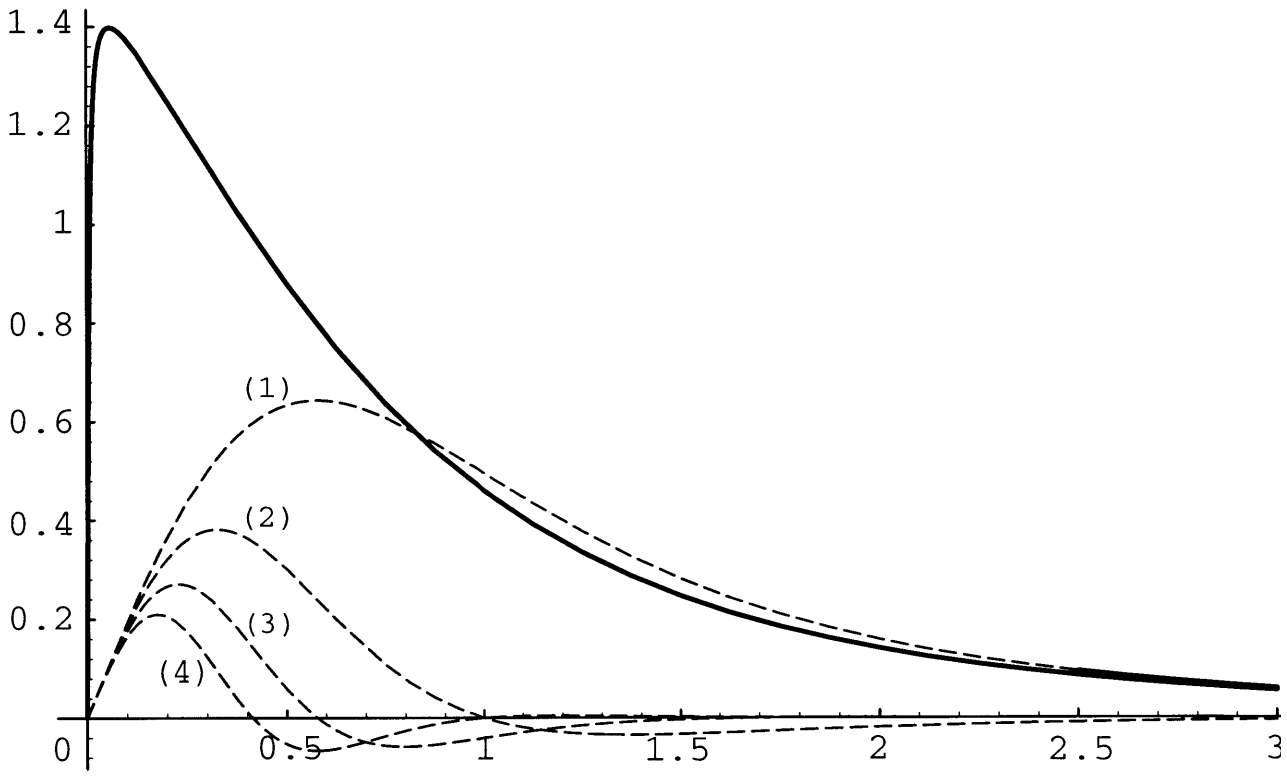

Figure 3: The phase shift for the Yukawa separable potential for $g = 0.99$. The solid curve is the exact result, the four dashed curves give the first four Born approximations respectively.

When $g > 1$ ($\lambda > \alpha^2$), it is no longer possible to use the Born approximation to evaluate the ω-integral. A glance at the second equation in eq. (52) confirms that the Born approximation cannot converge when there is a bound state: if it did, it would yield a result analytic in λ, however the ω-integral must generate a factor $\alpha\sqrt{\lambda}$ to cancel the contribution of the bound state and give $\mathcal{E}_c = -\frac{1}{2}\lambda$. To generate $\sqrt{\lambda}$, the Born expansion must not converge. Still, it is useful to subtract some number of terms in the Born expansion from the exact phase shift ($\bar{\delta}^N(\omega) = \delta(\omega) - \sum_{n=1}^{\infty} \lambda^n \delta^{(n)}(\omega)$) and to rewrite $\mathcal{E}_c$ accordingly,

$$\begin{aligned} \mathcal{E}_c = -\tfrac{1}{2}\lambda &= -\tfrac{1}{2}(\sqrt{\lambda} - \alpha)^2 - \frac{\lambda}{2\pi}\int_0^\infty d\omega\, \delta^{(1)}(\omega) - \frac{1}{2\pi}\int_0^\infty d\omega\, \bar{\delta}^N(\omega) \\ &= -\tfrac{1}{2}(\sqrt{\lambda} - \alpha)^2 - \tfrac{1}{2}\lambda - \frac{1}{2\pi}\int_0^\infty d\omega\, \bar{\delta}^N(\omega)\ . \end{aligned} \tag{56}$$

Here we have subtracted the first N terms in the Born approximation to convert δ to $\bar{\delta}^N$. When we added them back in, we used the fact that the integral of the first Born approximation gives exactly $-\frac{1}{2}\lambda$, and that the integrals of all higher Born approximations vanish. From this analysis we conclude that when there is a bound state, the integral of the "Born subtracted" phase shift, $\bar{\delta}^N$ gives an N-independent contribution to $\mathcal{E}_c$,

$$-\frac{1}{2\pi}\int_0^\infty d\omega\, \bar{\delta}^N(\omega) = \tfrac{1}{2}(\sqrt{\lambda} - \alpha)^2\ . \tag{57}$$

These results help to understand the effects that have been seen in numerical calculations in more complicated theories. In our toy model, when g is such that there is no bound state, the Casimir energy is given entirely by the first Born approximation (which, in a theory with divergences, would be computed from a Feynman graph). The n^{th} Born approximation gives a contribution to δ of order g^n, which may be large for $g <\approx 1$, but its integrated contribution to the Casimir energy is exactly zero. In more complicated theories the answer is very small but not exactly zero.[3] The magnitude of the effect can be seen in Fig. 3, where we plot the exact phase shift and the first four Born approximations for $g = 0.99$. The first Born approximation has the same integral as the full phase shift. Each higher Born approximation integrates to zero. The situation becomes even more bizarre in when there is a bound state. The entire Casimir energy is still given by the first Born approximation! The bound state contribution is exactly canceled by the remainder of the phase shift integral. One can remove any number of further Born approximations from δ – defining $\bar{\delta}^N$, which is significantly modified by the subtraction of higher Born approximations – without changing the integral. The effect can be seen in Fig. 4, where we plot the exact phase shift and the first four Born approximations for $g = 4.5$. Once again the toy model caricatures effects seen in more complex and more realistic theories.

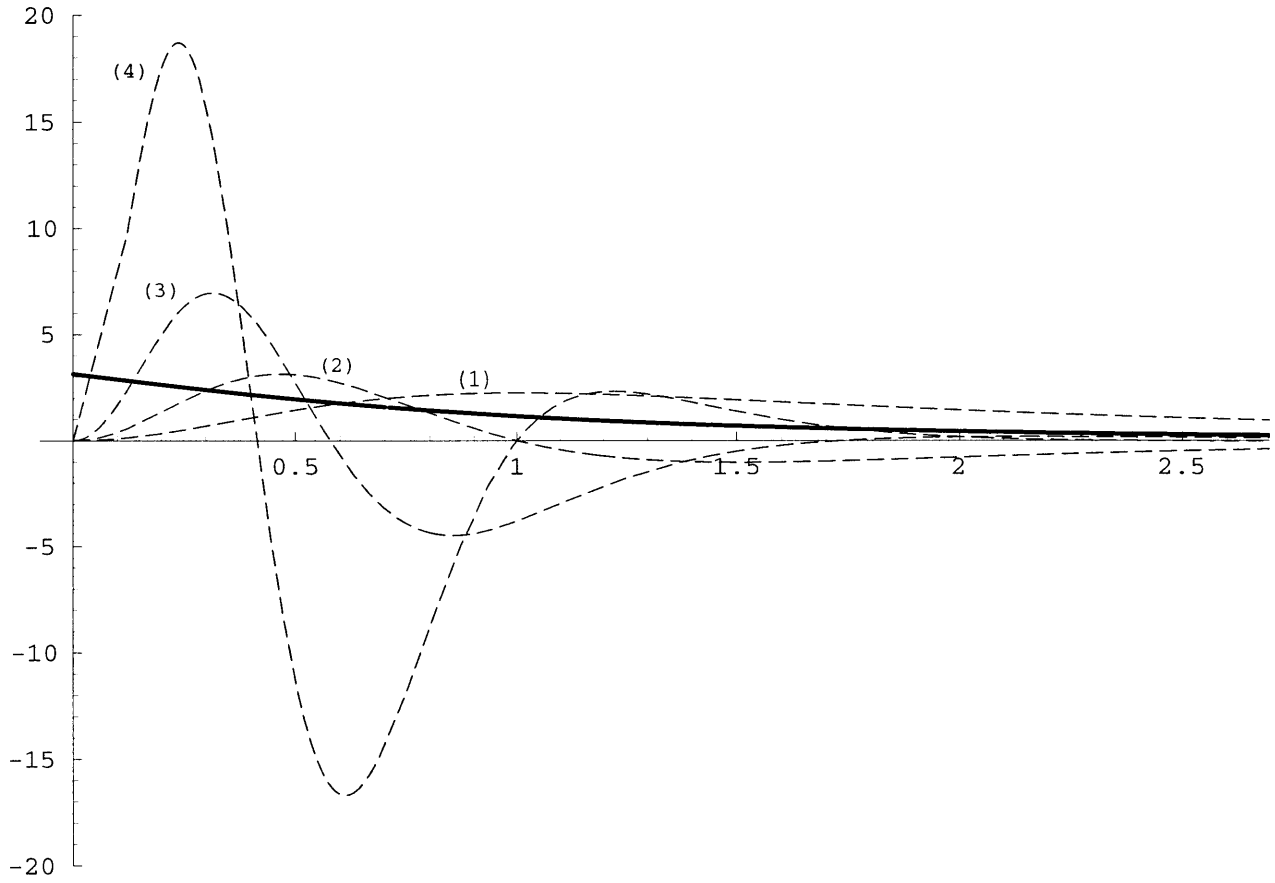

Figure 4: The phase shift for the Yukawa separable potential for $g = 4.5$. The solid curve is the exact result, the four dashed curves give the first four Born approximations respectively.

6 Summary and Conclusions

Our principal purpose has been to develop a simple model to obtain insight into complicated calculations of Casimir energies in quantum field theories. We have verified in our separable potential model that several traditional representations of the Casimir energy employed in quantum field theories are equivalent. Our model is very simple. In particular, it does not display the divergences characteristic of real quantum field theories. Thus we are not surprised that results are confirmed in our model that might be spoiled by divergences in more realistic theories. Nevertheless, the model is rich enough to illustrate some of the computational difficulties that have been seen numerically in more complicated theories.

Also, we should point out that there are cases where separable potentials do arise in the treatment of more realistic theories. In particular, Bashinsky has shown that the collective coordinate quantization in theories with solitons introduces additional, separable-potential-like terms into the small oscillations Hamiltonian whose eigenvalues determine the Casimir energy.[4] The ideas developed here would apply relatively directly to such cases.

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